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LETTER TO THE EDITOR

A new class of linearizable equations

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Abstract
Using the symmetry approach, we find a class of integrable nonlinear PDEs with dispersion law $\omega(k) = k^{\frac{3}{2}}$. All these equations turn out to be linearizable by means of a differential parametrization.

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1. Introduction

In this paper we consider integrable equations of the form

$$q_{tt} = q_{xxx} + F(q, q_x, q_t, q_{xx}, q_{xt}). \tag{1}$$

These equations possess an unusual dispersion law $\omega(k) = k^{\frac{3}{2}}$. Moreover, one can prove that for such equations there are no higher conserved densities, and therefore they are non-Hamiltonian.

The first example of such equations

$$q_{tt} = q_{xxx} + 3q_xq_{xt} + \left(q_t - 3q_x^2\right)q_{xx} \tag{2}$$

has been found in [1] (see also [12]). Some particular solutions of (2) feature moving branch-point singularities. For instance, the simplest self-similar solutions of the form $q = q(x + kt)$ satisfy the ODE

$$y' = y(y - k)^2 + \text{const}$$

where $q_x = y$. It is obvious that they have singularities of such type. This is in contradiction with the Painlevé test [4], an integrability criterion that asserts, roughly speaking, that integrable equations do not have solutions with such singularities.

Nevertheless, equation (2) possesses a degenerate Lax representation, implying the existence of infinitely many higher symmetries and corresponding invariant solutions [1].

Three examples of higher symmetries are

$$q_{t_1} = q_{xx} + 2q_x q_t - q_x^3$$
$$q_{t_2} = q_{xt} + q_x q_{xx} + q_t^2 + q_x^2 q_t - q_x^4$$
$$q_{t_3} = q_{tt} + 3q_x q_t^2 + 2q_{xx} q_t - 2q_x^3 q_t.$$

The whole hierarchy of symmetries can be reproduced by a recursion operator

$$\mathcal{R}_0 = -q_x + 2D^{-1} q_{xx} + D^{-1} D_t = A + B D_t \tag{3}$$

acting on the seed symmetries q_x and q_t. Since the properties of equation (2) seem to be rather unusual for standard integrable models, we decided to investigate in detail the whole class (1). The main goal of the paper is to find all equations in this class possessing higher symmetries, and to understand in what sense such equations are integrable.

In section 2 we generalize the main concepts of the symmetry approach [12], such as the formal recursion operator and canonical conserved densities, for the case of non-evolutionary equations of the form

$$q_{tt} = F(q, q_1, q_2, \ldots, q_n, q_t, q_{t1}, q_{t2}, \ldots, q_{tm}). \tag{4}$$

Equations of this type were excluded from consideration in works [2, 3, 5], where only evolution equations were investigated. Obviously, any equation (4) can be rewritten as a system of two evolution equations. For example, equation (2) is equivalent to

$$u_t = (v_x + uv)_x \qquad v_t = (u + v^2)_x$$

where $v = q_x$. However, the matrix coefficients of the leading derivatives of such systems have the structure of a Jordan block, whereas in papers [6–8], devoted to systems of evolution equations, the leading matrix was supposed to be diagonalizable. The standard method of deriving necessary integrability conditions based on the residues of fractional powers of formal recursion operator [12] does not work for such equations, and it has been necessary to develop a generalization of the methods of the symmetry approach. A new definition of the canonical densities proposed in this paper was inspired by the work [14]. Using these canonical densities, we find all equations (1) possessing infinitely many higher symmetries. It turns out (see section 3) that all these equations are related to second-order evolution equations

$$u_t = H(x, u, u_x, u_{xx}) \tag{5}$$

having higher symmetries. It was shown by Svinolupov [9] that any equation (5) can be reduced to one of the following equations:

$$u_t = u_{xx} + f(x)u$$
$$u_t = u_{xx} + 2uu_x + g(x)$$
$$u_t = \left(\frac{u_x}{u^2} + \lambda x\right)_x$$
$$u_t = \left(\frac{u_x}{u^2} + \lambda_1 x u + \lambda_2 u\right)_x$$

by a contact (or point) transformation

$$\bar{x} = \phi(x, u, u_x) \qquad \bar{u} = \psi(x, u, u_x) \tag{6}$$

where

$$\frac{\partial \phi}{\partial u_x}\left(\frac{\partial \psi}{\partial u} u_x + \frac{\partial \psi}{\partial x}\right) = \frac{\partial \psi}{\partial u_x}\left(\frac{\partial \phi}{\partial u} u_x + \frac{\partial \phi}{\partial x}\right).$$

All these equations can be linearized by simple differential substitutions (see [9]).

In section 3 we show that all equations (1) from our list admit a parametrization of the form

$$q_x = K(q, q_y, q_{yy}) \qquad q_t = S(q, q_y, q_{yy}, q_{yyy}) \tag{7}$$

where $q_x = K$ is a linearizable equation of second order, and $q_t = S$ is a higher symmetry of this latter equation. For example, the parametrization (7) of equation (2) is given by

$$q_x = \frac{q_{yy}}{q_y^2} \qquad q_t = -\frac{q_{yyy}}{q_y^3} + 3\frac{q_{yy}^2}{q_y^4}.$$

Any common solution $q(x, t, y)$ of system (7) gives a one-parameter family of solutions of the integrable equation (1). However, the general solution of (7) depends on one function $q_0(y) = q(0, 0, y)$ of one variable, whereas the general solution of (1) depends on two functions of one variable.

Although a general idea of such parametrization is contained in [1, 11], explicit formulae for the linearization of equation (2) were first obtained by Adler [10].

2. Classification of third-order equations

2.1. Integrability conditions

For equation (4), all the mixed derivatives of q containing at least two time derivatives can be expressed in terms of

$$q, q_x, q_{xx}, \ldots, q_i, \ldots \qquad q_t, q_{t1} = q_{tx}, q_{t2} = q_{txx}, \ldots, q_{ti}, \ldots \tag{8}$$

in virtue of (4). The derivatives (8) will be regarded as *independent* variables.

An equation

$$q_\tau = G(q, q_1, q_2, \ldots, q_r, q_t, q_{t1}, q_{t2}, \ldots, q_{ts}) \tag{9}$$

compatible with (4) is called infinitesimal (local) symmetry of (4). Compatibility implies that the function G satisfies the equation $\mathcal{F}(G) = 0$, where

$$\mathcal{F} = D_t^2 - \sum_{i=0}^{n} \frac{\partial F}{\partial q_i} D_x^i - \left(\sum_{i=0}^{m} \frac{\partial F}{\partial q_{ti}} D_x^i \right) D_t \stackrel{\text{def}}{=} D_t^2 - (M + N D_t) \tag{10}$$

is the linearization operator for equation (4).

In order to rewrite the consistency conditions of (4) and (9) in terms of a series of conservation laws

$$(\rho_i)_t = (\sigma_i)_x \tag{11}$$

for (4), one can use a formal Lax representation of the problem. The linearization of equations (4) and (9) gives rise to the compatibility problem for linear equations

$$\phi_{tt} = (M + N D_t)\phi \qquad \phi_\tau = (A + B D_t)\phi$$

or, equivalently,

$$\Phi_t = F_* \Phi \qquad \Phi_\tau = G_* \Phi \qquad \Phi = \begin{pmatrix} \phi \\ \phi_t \end{pmatrix} \qquad F_* = \begin{pmatrix} 0 & 1 \\ M & N \end{pmatrix}$$

where

$$G_* = \begin{pmatrix} A & B \\ \hat{A} & \hat{B} \end{pmatrix} \qquad \hat{A} \stackrel{\text{def}}{=} A_t + BM \qquad \hat{B} \stackrel{\text{def}}{=} B_t + BN + A. \tag{12}$$

Cross differentiation yields

$$D_t(G_*) = [F_*, G_*] + D_\tau(F_*) \tag{13}$$

where F_*, G_* are matrix differential operators. The crucial step in the symmetry approach (see [3, 6, 12] and references there) is to consider instead of equation (13), the equation

$$D_t(R) = [F_*, R] \tag{14}$$

where R is matrix pseudo-differential operator. We call R a *matrix formal recursion operator.*

Denoting as before $R_{11} = A$, $R_{12} = B$ we can rewrite (14) as

$$A_{tt} - NA_t + [A, M] + (2B_t + [B, N])M + BM_t = 0 \tag{15}$$

$$B_{tt} + 2A_t + [B, M] + [A, N] + ([B, N] + 2B_t)N + BN_t - NB_t = 0. \tag{16}$$

If R_1, R_2 are formal matrix recursion operators, then $R_3 = R_1R_2$ is also a formal recursion operator, and we find using (12) that

$$A_3 = A_1A_2 + B_1B_2M + B_1A_{2,t} \qquad B_3 = A_1B_2 + B_1A_2 + B_1B_2N + B_1B_{2,t}. \tag{17}$$

The identities (15) and (16) mean that the scalar pseudo-differential operator $\mathcal{R} = A + BD_t$ is related to (10), the linearization $\mathcal{F}$ of equation (4), by

$$\mathcal{F}(A + BD_t) = (\bar{A} + BD_t)\mathcal{F} \tag{18}$$

where $\bar{A} = A + 2B_t + [B, N]$. A pseudo-differential operator $\mathcal{R} = A + BD_t$, with components

$$A = \sum_{-\infty}^{n} a_i D_x^i \qquad B = \sum_{-\infty}^{m} b_i D_x^i$$

satisfying (15) and (16), is called *scalar formal recursion operator* for equation (4). If A and B are differential operators (or ratios of differential operators), condition (18) implies that the operator $\mathcal{R}$ maps symmetries of equation (4) again to symmetries. However, we are using the notion of formal recursion operator with a completely different aim.

Let $\mathcal{R}_1 = A_1 + B_1D_t$ and $\mathcal{R}_2 = A_2 + B_2D_t$ be two scalar formal recursion operators. Then the product $\mathcal{R}_3 = \mathcal{R}_1\mathcal{R}_2$, in which D_t^2 is replaced by $(M + ND_t)$, is also a scalar formal recursion operator, with components given by (17).

An operator $\mathcal{S} = P + QD_t$ is said to be implectic if

$$\mathcal{F}^*\mathcal{S} + \bar{\mathcal{S}}\mathcal{F} = 0 \qquad \bar{\mathcal{S}} = \bar{P} + \bar{Q}D_t.$$

Here and in what follows the superscript '$*$' denotes the adjoint operator. If $\mathcal{S}$ can be applied to symmetries, then it maps symmetries to cosymmetries. In the symmetry approach P and Q are supposed to be formal non-commutative series with respect to D_x.

The operator equations for the components of the formal implectic operator $\mathcal{S} = P + QD_t$ have the following form:

$$P_{tt} + N^*P_t + 2Q_tM + QM_t = M^*P - PM - (QN + N^*Q)M - N_t^*P \tag{19}$$

$$\begin{aligned} Q_{tt} + 2P_t + 2Q_tN + N^*Q_t = {} & M^*Q - QM - (QN + N^*Q)N \\ & - PN - N^*P - (N_t^*Q + QN_t). \end{aligned} \tag{20}$$

The linearization $P + QD_t$ of the variational derivative of a conserved density for equation (4) satisfies equations (19) and (20) up to a 'small' rest (see [13]).

Let us consider equations of form (1). It follows from formulae (19) and (20) that equation (1) has no higher conservation laws. Moreover, it is easy to prove that the density of any conservation law, up to total derivatives, is of the form

$$\rho = r_1(q, q_x)q_t + r_2(q, q_x).$$

All statements presented below can be easily reformulated for general equations of type (4).

Theorem 1. *If equation (1) possesses an infinite sequence of higher symmetries of the form*

$$q_{\tau_i} = G_i\big(q, q_1, q_2, \ldots, q_{r_i}, q_t, q_{t1}, q_{t2}, \ldots, q_{ts_i}\big) \tag{21}$$

then there exists a formal recursion operator of the form

$$\mathcal{R} = (a_0 + a_{-1}D^{-1} + \cdots) + (D^{-1} + b_{-2}D^{-2} + \cdots)D_t \tag{22}$$

where a_i, b_i are some functions of the variables (8).

For scalar evolution equations one can use (see [3, 6, 12]) the residues of powers of the formal recursion operator to derive the canonical conservation laws (11). Unfortunately, for equations (1) this technique does not work, and we present a different way (cf [14]) to get necessary integrability conditions (11).

Let $\mathcal{R}$ be a formal recursion operator of form (22). It is then possible to find an operator

$$\mathcal{R}^{-1} = (\alpha_{-1}D^{-1} + \alpha_{-2}D^{-2}\cdots) + (D^{-2} + \beta_{-3}D^{-3} + \cdots)D_t \tag{23}$$

such that $\mathcal{R}\mathcal{R}^{-1} = \mathcal{R}^{-1}\mathcal{R} = 1$. Recall that we eliminate D_t^2 in virtue of $\mathcal{F} = 0$ in the product of scalar recursion operators. The operator $\mathcal{R}^{-1}$ is uniquely defined.

Theorem 2. *If $\mathcal{R}$ is a formal recursion operator of form (22) for equation (1), then there is a unique representation of the total derivative operators D_x and D_t of the form*

$$D_x = \sum_{-\infty}^{2} \rho_i \mathcal{R}^i \qquad D_t = \sum_{-\infty}^{3} \sigma_i \mathcal{R}^i. \tag{24}$$

Functions ρ_i and σ_i are densities and fluxes of some (maybe trivial) conservation laws (11) for equation (1).

The following formulae define five integrability conditions (11) for equations (1):

$$\rho_1 = \frac{\partial F}{\partial q_{xt}}$$

$$\rho_2 = \frac{\partial F}{\partial q_{xx}} + \frac{2}{3}\sigma_1$$

$$\rho_3 = 6\sigma_2 - \frac{\partial F}{\partial q_{xt}}\sigma_1 + 9\frac{\partial F}{\partial q_t} - 3\frac{\partial F}{\partial q_{xt}}\frac{\partial F}{\partial q_{xx}} - \frac{1}{3}\frac{\partial F}{\partial q_{xt}}^3$$

$$\rho_4 = 6\sigma_3 - 9\frac{\partial F}{\partial q_{xt}}\sigma_2 + 3\sigma_1^2 + 27\frac{\partial F}{\partial q_t}\frac{\partial F}{\partial q_{xt}} - \frac{\partial F}{\partial q_{xt}}^4 + 81\frac{\partial F}{\partial q_x} - 9\frac{\partial F}{\partial q_{xt}}^2\frac{\partial F}{\partial q_{xx}} - 27\frac{\partial F}{\partial q_{xx}}^2$$

$$\rho_5 = 2\sigma_4 + 18\sigma_1\sigma_2 - 27(\sigma_1)_t - 3\sigma_1^2\frac{\partial F}{\partial q_{xt}} - 3\sigma_3\frac{\partial F}{\partial q_{xt}} - 9\sigma_1\frac{\partial F}{\partial q_{xt}}\frac{\partial F}{\partial q_{xx}} - \sigma_1\frac{\partial F}{\partial q_{xt}}^3 + 27\sigma_1\frac{\partial F}{\partial q_t}.$$

The conditions mean that ρ_i are densities of *local* conservation laws for equation (1). In other words, for any ρ_i there exists a corresponding function σ_i depending on the variables (8).

2.2. List of integrable equations

Lemma. *If equation (1) satisfies conditions (11) with $i = 1, 2, 3, 4$, then it is of the form*

$$q_{tt} = q_{xxx} + (A_1 q_t + A_2) q_{xt} + A_3 q_{xx}^2 + \left(A_4 q_t^2 + A_5 q_t + A_6\right) q_{xx} + A_7 q_t^4 + A_8 q_t^3 + A_9 q_t^2 + A_{10} q_t + A_{11} \tag{25}$$

where the functions A_i depend on q and q_x only.

It is easy to verify that the class of equations (25) is invariant with respect to point transformations $q \to \varphi(q)$. Moreover, if all functions A_i do not depend on q, then shifts of the form $q \to q + \lambda_1 x + \lambda_2 t$, where λ_i are arbitrary constants, are also allowed.

Theorem 3. *Up to the transformations described above, any nonlinear equation (1) satisfying integrability conditions with $i = 1, 2, \ldots, 7$, coincides with the equation*

$$q_{tt} = q_{xxx} + (3q_x + k) q_{xt} + \left(q_t - 3q_x^2 - 2kq_x + 6\wp\right) q_{xx} - 2\wp' q_t + 6\wp' q_x^2 + (\wp'' + k\wp') q_x \tag{26}$$

where $\wp(q)$ is any solution of an equation of the form

$$\wp'^2 = 8\wp^3 + k^2\wp^2 + c_1\wp + c_0 \tag{27}$$

or with the equation

$$q_{tt} = q_{xxx} + \left(\frac{3q_t}{q_x} + \frac{3}{2} X(q)\right) q_{xt} - \frac{1}{q_x} q_{xx}^2 - \left(\frac{2q_t^2}{q_x^2} + \frac{3q_t}{2q_x} X(q)\right) q_{xx} + c_2 \left(q_x q_t + \frac{3}{2} q_x^2 X(q)\right) \tag{28}$$

where $X(q) = c_2 q + c_1$ and c_i are arbitrary constants.

Remark. Actually, any equation (28) can be reduced to the equation with $X(q) = \text{const}$ or to the equation with $X(q) = q$. The integrability conditions (6) and (7) have quite long expressions and we do not present them in this paper. We have only used these conditions to prove that any equation (1) for which all conservation laws (11) are trivial, is equivalent to a linear one.

2.3. Recursion operators

In this section we present a closed form for recursion operators of the models (26) and (28). The existence of these recursion operators implies the fact that all the integrability conditions with $i \geqslant 1$ are fulfilled for these equations.

Let us consider equation (26). This equation has only one non-trivial conserved density, given by

$$\rho = q_t - q_x^2 + 2\wp(q). \tag{29}$$

The simplest higher symmetries of (26) are

$$q_\tau = q_{xx} + 2q_x q_t - q_x^3 - kq_x^2 + 4\wp q_x$$

and

$$q_\tau = q_{xt} + q_x q_{xx} + q_t^2 + \left(q_x^2 + 2\wp\right) q_t - q_x^4 - kq_x^3 + 6\wp q_x^2 + (\wp' + k\wp) q_x.$$

They are generated by the following recursion operator:

$$\mathcal{R} = D + \left(q_t - 2q_x^2 - kq_x + 2\wp\right) + q_x D^{-1} (D_t + 2q_{xx} + 2\wp')$$

acting on the seed symmetries q_x and q_t. A direct calculation shows that this recursion operator satisfies (15) and (16). In the degenerate case $\wp \equiv 0, k = 0$ we have $\mathcal{R} = \mathcal{R}_0^2$, where $\mathcal{R}_0$ is defined by (3). The operator $D_t + 2q_{xx} + 2\wp'$ corresponds to the variational derivative of function (29) and, therefore, if we apply $\mathcal{R}$ to any local symmetry admitting the conservation law with density (29), the result should be local.

Another recursion operator for (26) has the form

$$\mathcal{S} = D_t + \left(q_{xx} - q_x^3 - kq_x^2 + 6\wp q_x + k\wp + \wp'\right) + q_t D^{-1}(D_t + 2q_{xx} + 2\wp').$$

One can verify that

$$\mathcal{S}^2 = \mathcal{R}^3 - k\mathcal{R}\mathcal{S} - \frac{c_1}{2}\mathcal{R} - c_0.$$

For equation (28), a recursion operator is given by

$$\mathcal{R} = \left(\frac{q_t}{q_x} + \frac{1}{2}X\right) - q_x D^{-1}\left(\frac{1}{q_x}D_t + \frac{q_{xt}}{q_x^2} - \frac{2q_t q_{xx}}{q_x^3} + \frac{c_2}{2}\right).$$

The non-trivial conserved density for this equation is given by

$$\rho = \frac{q_t}{q_x} + \frac{1}{2}X(q). \tag{30}$$

In the case $X(q) = 0, c_1 = 0$ the non-local variable $Q = D_x^{-1}(\rho)$ satisfies equation (2).

3. Linearization procedure

Both equations (26) and (28) have only one non-trivial conserved local density. The Burgers equation $u_t = u_{xx} + 2uu_x$ possesses the same property: $D_t(u) = D_x(u_x + u^2)$ is the unique conservation law of this equation. The crucial step in the linearization of the Burgers equation is to introduce the potential W of this conservation law. By definition, the variable W satisfies conditions $W_x = u$, $W_t = u_x + u^2$. It is easy to verify that the function $U = \exp(W)$ satisfies the heat equation $U_t = U_{xx}$.

For equations (26) and (28) the potentials of conservation laws also play a key role in linearization. However, the procedure of linearization is not so straightforward. To illustrate it, let us consider the simplest version, $X = 0$, of equation (28). The potential of the conservation law for this equation satisfies the conditions

$$W_x = \frac{q_t}{q_x} \qquad W_t = \frac{q_{xx}}{q_x} + \frac{q_t^2}{q_x^2}.$$

A simple computation shows that the equation admits the non-local symmetry $q_y = W$. Since

$$D_y\left(\frac{q_t}{q_x}\right) = \frac{q_{xx}}{q_x^2}$$

we have

$$q_{yy} = W_y = -\frac{1}{q_x} \qquad q_{yyy} = \frac{q_t}{q_x^3}$$

or

$$q_x = -\frac{1}{q_{yy}} \qquad q_t = -\frac{q_{yyy}}{q_{yy}^3}.$$

After the Legendre transformation

$$y = U_z \qquad q = U - zU_z$$

the latter pair of compatible equations becomes linear:

$$U_x = U_{zz} \qquad U_t = U_{zzz}.$$

For the more complicated equation

$$q_{tt} = q_{xxx} + \frac{3q_t}{q_x} q_{xt} - \frac{1}{q_x} q_{xx}^2 - \frac{2q_t^2}{q_x^2} q_{xx} + c\left(q_{xt} - \frac{q_t}{q_x} q_{xx}\right) \tag{31}$$

which corresponds to the case $X = \text{const} \neq 0$, the same linearization scheme works. The potential is defined by

$$W_x = \frac{q_t}{q_x} \qquad W_t = \frac{q_{xx}}{q_x} + \frac{q_t^2}{q_x^2} + c\frac{q_t}{q_x}$$

and the non-local symmetry is given by $q_y = \exp(-cW)$. It is not difficult to check that

$$q_x = -c^2 \frac{q_y^2}{q_{yy}} \qquad q_t = c^3 \frac{q_y^3 q_{yyy} - 2q_y^2 q_{yy}^2}{q_{yy}^3}.$$

After a contact transformation

$$y = \tfrac{1}{2}\exp(-z)(U_z + U) \qquad q = \tfrac{1}{2}\exp(z)(U_z - U)$$

these equations become

$$U_x = \frac{c^2}{2}(U_{zz} - U) \qquad U_t = \frac{c^3}{4}(U_{zzz} + U_{zz} - U_z - U).$$

The most non-trivial case is $X = q$. In this case the potential W is defined by

$$W_x = \frac{q_t}{q_x} + \frac{q}{2} \qquad W_t = \frac{q_{xx}}{q_x} + \frac{q_t^2}{q_x^2} + \frac{3q_t}{2q_x} q + \frac{3}{4} q^2$$

but no non-local symmetries of the form $q_y = F(q, W)$ exist. However, there exists a new non-local conservation law with potential Z defined by

$$Z_x = q^2 - 2Wq_x \qquad Z_t = -\tfrac{1}{2}q^3 - 2q_x - 2Wq_t.$$

Using these two potentials, we find a non-local symmetry $q_y = 2q\exp\left(-\frac{Z}{4} - \frac{qW}{2}\right)$. Expressing q_x and q_t in terms of the y-derivative, we get

$$q_x = -\frac{q^3 q_y^2}{4(qq_{yy} - 2q_y^2)} \qquad q_t = \frac{q^4 q_y^3 (q^2 q_{yyy} - 9qq_y q_{yy} + 12q_y^3)}{8(qq_{yy} - 2q_y^2)^3}.$$

After the contact transformation $y = z + \frac{U}{U_z}$, $q = -\frac{U_z}{U^2}$ we do not obtain linear equations but

$$U_x = \frac{1}{4} D_z \left(\frac{U_z}{U^2}\right) \qquad U_t = -\frac{1}{8} D_z \left(\frac{U_{zz}}{U^3} - \frac{3U_z^2}{U^4}\right).$$

To linearize this system one can introduce a potential Y such that $Y_z = U$, $Y_x = \frac{U_z}{4U^2}$, and after that make a point transformation $Y \leftrightarrow z$.

Equation (26) can be linearized as follows. It is easy to verify that it has a non-local symmetry $q_y = A(q)\exp(-W)$, where

$$W_x = q_t - q_x^2 + 2\wp \qquad W_t = q_{xx} - q_x^3 + q_t q_x + k(q_t - q_x^2 + \wp) + 6\wp q_x + \wp' + w$$

and

$$B'^2 = B^4 + \frac{k^2}{2} B^2 + 8wB + b_0 \qquad B = -\frac{A'}{A} + \frac{k}{2}.$$

It can be checked that

$$q_x = \frac{q_{yy}}{q_y^2} + 2B(q) \tag{32}$$

$$q_t = -\frac{q_{yyy}}{q_y^3} + 3\frac{q_{yy}^2}{q_y^4} + 3\frac{q_{yy}}{q_y^2}B(q) + \frac{k}{2}\left(\frac{q_{yy}}{q_y^2} + 2B(q)\right) - \frac{3}{2}(B'(q) - B(q)^2) - \frac{k^2}{8}. \tag{33}$$

The function $\wp$ from equation (26) is given by

$$\wp = -\frac{1}{4}B' + \frac{1}{4}B^2 - \frac{k^2}{48}$$

where the parameters of the elliptic functions $\wp$ and B are related by

$$c_1 = \frac{k^4 - 16b_0}{32} \qquad c_0 = w^2.$$

After a change of variables $z = q, u = y$, equations (32) and (33) take the following linear form:

$$u_x = u_{zz} - 2B(z)u_z$$
$$u_t = -u_{zzz} + 3B(z)u_{zz} + \frac{3}{2}(B'(z) - B(z)^2)u_z + \frac{k}{2}(u_{zz} - 2B(z)u_z) + \frac{k^2}{8}u_z.$$

We see that in all cases there exists a non-local symmetry $q_y = G$, depending on the potentials, such that q_x and q_t can be expressed in terms of y-derivatives by formulae (7). Using this parametrization, one can construct particular solutions of equations (26) and (28).

A parametrization of such kind arises not only for linearizable equations, but also for equations of KdV-type and associated linear spectral problems. For example, consider the spectral problem

$$\Psi_{xx} = (\lambda^3 + u_1\lambda^2 + u_2\lambda + u_3)\Psi. \tag{34}$$

For a non-local symmetry of this linear equation one can take (see [1, 11])

$$\bar{\Psi}_{yy} = \frac{\lambda}{a^2}\bar{\Psi} \qquad \bar{\Psi} = \frac{1}{\sqrt{a}}\Psi.$$

Then

$$u_1 = \frac{1}{4}a_y^2 - \frac{1}{2}aa_{yy} \qquad u_2 = -a[\log(a)]_{xy} \qquad u_3 = -\frac{a_{xx}}{2a} + \frac{3a_x^2}{4a^2}$$

where $a(x, y)$ satisfies the Harry–Dym equation

$$a_x = a^3 a_{yyy}.$$

At least on the local level, the general solution of the latter equation depends on three arbitrary functions $a_0(x) = a(x, 0)$, $a_1(x) = a_y(x, 0)$, $a_2(x) = a_{yy}(x, 0)$ of x, and therefore this parametrization provides a generic potential in (34). Multi-phase solutions of the Harry–Dym equation lead to special potentials of the spectral problem (34).

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